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Damour–Ruffini and Unruh Theories of the Hawking Effect

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Internal relations between the Damour–Ruffini approach and the Unruh approach to dealing with the Hawking effects are shown. The Unruh-type analytical wave functions can be obtained by means of the analytical continuation method suggested by Damour and Ruffini. In fact, Unruh-type analytical wave functions are complex conjugate functions of Damour–Ruffini type. Normalizing each of them, or making use of them to construct the Bogoliubov transformation, one can get the same Hawking thermal spectrum. The pure state wave function defined on the connected complex r space-time manifold is a mixture showing thermal properties in the real r space-time manifold, which is divided into two parts by the event horizon.

Different approaches to verifying the Hawking effect have been suggested by Hawking (1975), Unruh (1976), Hartle and Hawking (1976), Gibbons and Hawking (1977), Damour and Ruffini (1976), and others. The Hawking, Unruh, Hartle, and Gibbons approaches are often used in black hole thermodynamics. There has been recent interest in the Damour–Ruffini approach (Zhao and Gui, 1983; Sannan, 1988; Kim *et al.*, 1989) and it has been successfully applied to deal with the Hawking effect of nonstationary black holes (Zhao and Dai, 1992; Zhao, 1993). The purpose of this paper is to point out the internal relation of the Damour–Ruffini approach and the Unruh approach, and to explore the physical aspect of the Damour–Ruffini approach.

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1. DAMOUR–RUFFINI APPROACH

After separation of variables as (Damour and Ruffini, 1976)

$$\Phi_{\omega lm} = \frac{1}{(4\pi\omega)^{1/2}} \frac{1}{r} R(r, t) Y_{lm}(\theta, \phi) \tag{1}$$

the Klein–Gordon equation

$$(g^{\mu\nu}\nabla_\mu \nabla_\nu - \mu^2)\Phi = 0 \tag{2}$$

in the Schwartzschild space-time

$$ds^2 = -(1 - 2M/r)\, dt^2 + (1 - 2M/r)^{-1}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3}$$

can be written as a radial equation and an angular equation,

$$\left[-\left(1 - \frac{2M}{r}\right)^{-1} r^2 \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial r}(r^2 - 2Mr)\frac{\partial}{\partial r} + \mu^2 r^2\right]\frac{R}{r} = -\frac{l(l+1)R}{r} \tag{4}$$

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)Y_{lm} = l(l+1)Y_{lm} \tag{5}$$

where M is the mass of the black hole, and μ, ω, l, and m are, respectively, the mass, energy, angular momentum quantum number, and magnetic quantum number of the KG particle. With the tortoise coordinate

$$r_* = r + (1/2\kappa)\ln|r - r_H| = \begin{cases} r + (1/2\kappa)\ln(r - r_H), & r > r_H \\ r + (1/2\kappa)\ln(r_H - r), & r < r_H \end{cases} \tag{6}$$

we can write equation (4) as

$$\left\{-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - \left(1 - \frac{2M}{r}\right)\left[\frac{2M}{r^3} + \frac{l(l+1)}{r^2} + \mu^2\right]\right\}R(r_*, t) = 0 \tag{7}$$

where $r_H = 2M$ and $\kappa = 1/4M$ are, respectively, the location and the surface gravity of the event horizon. Let r go to r_H; equation (7) can be reduced to

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2}\right)R = 0 \tag{8}$$

so we obtain the ingoing wave solution

$$R^{\text{in}} = e^{-i\omega(t + r_*)} = e^{-i\omega v} \tag{9}$$

and the outgoing wave solution

$$R_{\text{I}}^{\text{out}} = e^{-i\omega(t - r_*)} = e^{-i\omega u} \qquad (r > r_H) \tag{10}$$

outside the event horizon and near it. Here

$$v = t + r_*, \qquad u = t - r_* \tag{11}$$

are, respectively, the advanced and the retarded Eddington–Finkelstein coordinates. When we adopt the advanced coordinate v, the components of the metric tensor are analytical at the event horizon. The ingoing wave is also analytical. But the outgoing wave

$$R_{\rm I}^{\rm out} = e^{2i\omega r_*} e^{-i\omega v} = e^{2i\omega r} e^{-i\omega v}(r - 2M)^{i4M\omega} \qquad (r > r_H) \tag{12}$$

is not. Damour and Ruffini extend it by analytical continuation to the inside of the horizon through the lower half complex r plane:

$$(r - 2M) \to |r - 2M|\, e^{-i\pi} = (2M - r)\, e^{-i\pi} \tag{13}$$

$$R_{\rm I}^{\rm out} \to \tilde{R}_{\rm II}^{\rm out} = e^{4\pi M\omega} e^{2i\omega r_*} e^{-i\omega v} \qquad (r < r_H) \tag{14}$$

Then we get the general outgoing wave

$$\phi_\omega = N_\omega (R_{\rm I}^{\rm out} + e^{4\pi M\omega} R_{\rm II}^{\rm out*}) \tag{15}$$

where N_ω is a normalization factor, and

$$R_{\rm I}^{\rm out} = \begin{cases} e^{2i\omega r_*} e^{-i\omega v}, & r > r_H \\ 0, & r < r_H \end{cases} \tag{16}$$

$$R_{\rm II}^{\rm out*} = \begin{cases} e^{2i\omega r_*} e^{-i\omega v}, & r < r_H \\ 0, & r > r_H \end{cases} \tag{17}$$

The inner product is

$$(\phi_\omega, \phi_\omega) = N_\omega^2 (1 - e^{8\pi M\omega}) \tag{18}$$

Because $e^{8\pi M\omega} > 1$, we have

$$(\phi_\omega, \phi_\omega) = -1 \tag{19}$$

so

$$N_\omega^2 = \frac{1}{e^{\omega/k_{\rm B}T} - 1} \tag{20}$$

where

$$T = \frac{\kappa}{2\pi k_{\rm B}} \tag{21}$$

This is the Hawking temperature.

2. FROM DAMOUR–RUFFINI TO UNRUH

Solving the wave equation (8) respectively inside and outside the event horizon, we find that both of the ingoing wave solutions are the same

$$R^{\rm in} = e^{-i\omega(t + r_*)} = e^{-i\omega v} \tag{22}$$

It is analytical at the event horizon. The outgoing wave solutions are, respectively,

$$R_{\rm I}^{\rm out} = e^{-i\omega(t-r_*)} = e^{-i\omega u} \qquad (r > r_H) \tag{23}$$

$$R_{\rm II}^{\rm out} = e^{-i\omega(r_*-t)} = e^{i\omega u} \qquad (r < r_H) \tag{24}$$

Here, r_* is a time coordinate, and t is a space coordinate inside the black hole. Equations (23) and (24) are analytical respectively outside and inside the horizon. But neither is analytical at the horizon. The field modes (23) and (24) form a complete orthonormal basis for outgoing waves. Every outgoing wave packet crossing the event horizon may be expanded by them as

$$\phi^{\rm out} = \int d\omega [b_\omega^{(1)} R_{\rm I}^{\rm out} + b_\omega^{(2)} R_{\rm II}^{\rm out} + b_\omega^{(1)\dagger} R_{\rm I}^{\rm out*} + b_\omega^{(2)\dagger} R_{\rm II}^{\rm out*}] \tag{25}$$

where $b_\omega^{(1)}$, $b_\omega^{(2)}$, $b_\omega^{(1)\dagger}$, and $b_\omega^{(2)\dagger}$ are, respectively, annihilation operators and creation operators of the particles outside and inside the event horizon. With them, one can have a Fock basis and define the outgoing vacuum state $|0\rangle_{\rm out}$ as

$$b_\omega^{(1)}|0\rangle_{\rm out} = b_\omega^{(2)}|0\rangle_{\rm out} = 0 \qquad \forall\omega \tag{26}$$

On the other hand, extending the outgoing wave (23) by analytical continuation to the inside of the event horizon through the lower half complex r plane, one gets another wave function

$$\tilde{R}_{\rm II}^{\rm out} = e^{4\pi M\omega}\, e^{-i\omega u} = e^{\pi\omega/\kappa} R_{\rm II}^{\rm out*} \qquad (r < r_H) \tag{27}$$

where $\kappa = 1/4M$. This is the same as equation (14). $R_{\rm II}^{\rm out*}$ is the complex conjugate function of (24). In fact, it is just equation (17).

Introducing the null Kruskal coordinates as

$$\begin{cases} U = -(1/\kappa)\, e^{-\kappa u}, \\ V = (1/\kappa)\, e^{\kappa v} \qquad (r > r_H) \end{cases} \tag{28}$$

$$\begin{cases} U = (1/\kappa)\, e^{-\kappa u} \\ V = (1/\kappa)\, e^{\kappa v} \qquad (r < r_H) \end{cases} \tag{29}$$

or

$$\begin{cases} u = (-1/\kappa)\ln(-\kappa U) \\ v = (1/\kappa)\ln(\kappa V) \qquad (r > r_H) \end{cases} \tag{30}$$

$$\begin{cases} u = (-1/\kappa)\ln(\kappa U) \\ v = (1/\kappa)\ln(\kappa V) \qquad (r < r_H) \end{cases} \tag{31}$$

One can rewrite equations (23), (24) and (27), respectively, as

$$\begin{aligned} R_{\mathrm{I}}^{\mathrm{out}} &= \exp[(i\omega/\kappa)\ln(-\kappa U)] \qquad (r > r_H) \\ R_{\mathrm{II}}^{\mathrm{out}} &= \exp[(-i\omega/\kappa)\ln(\kappa U)] \qquad (r < r_H) \\ \tilde{R}_{\mathrm{II}}^{\mathrm{out}} &= \exp(\pi\omega/\kappa)\exp[(i\omega/\kappa)\ln(\kappa U)] \\ &= \exp[(i\omega/\kappa)\ln(-\kappa U)] \end{aligned} \tag{32}$$

Here, we used the relation

$$\ln(-1) = -i\pi \tag{33}$$

The functional form of $R_{\mathrm{I}}^{\mathrm{out}}$ is different from that of $R_{\mathrm{II}}^{\mathrm{out}}$. But it is the same as that of $\tilde{R}_{\mathrm{II}}^{\mathrm{out}}$. Therefore, the combination of $R_{\mathrm{I}}^{\mathrm{out}}$ and $\tilde{R}_{\mathrm{II}}^{\mathrm{out}}$ can be regarded as an identical and analytical wave function, crossing the event horizon, as

$$\rho_1 \sim R_{\mathrm{I}}^{\mathrm{out}} + \tilde{R}_{\mathrm{II}}^{\mathrm{out}} = R_{\mathrm{I}}^{\mathrm{out}} + e^{\pi\omega/\kappa} R_{\mathrm{II}}^{\mathrm{out}*} \tag{34}$$

Similarly, if one extends the complex conjugate of the outgoing wave $R_{\mathrm{I}}^{\mathrm{out}}$

$$R_{\mathrm{I}}^{\mathrm{out}*} = e^{i\omega u} \tag{35}$$

by analytical continuation to the inside of the horizon through the lower half complex r plane, one has

$$\tilde{R}_{\mathrm{II}}^{\mathrm{out}*} = e^{-\pi\omega/\kappa}\, e^{i\omega u} = e^{-\pi\omega/\kappa} R_{\mathrm{II}}^{\mathrm{out}} \tag{36}$$

Clearly, the combination

$$\rho_2 \sim R_{\mathrm{I}}^{\mathrm{out}*} + \tilde{R}_{\mathrm{II}}^{\mathrm{out}*} = R_{\mathrm{I}}^{\mathrm{out}*} + e^{-\pi\omega/\kappa} R_{\mathrm{II}}^{\mathrm{out}} \tag{37}$$

can also be regarded as an identical and analytical wave function, crossing the horizon.

In fact, equation (34) is the same as equation (15) except for the normalization factor N_ω. Calculating their inner product, one has

$$\begin{aligned} (\rho_1, \rho_1) &\sim (1 - e^{2\pi\omega/\kappa}) < 0 \\ (\rho_2, \rho_2) &\sim (-1 + e^{-2\pi\omega/\kappa}) < 0 \end{aligned} \tag{38}$$

so both ρ_1 and ρ_2 are negative-frequency waves. We define

$$\begin{aligned} f_1^* &= e^{\pi\omega/2\kappa}\rho_2 = e^{\pi\omega/2\kappa} R_{\mathrm{I}}^{\mathrm{out}*} + e^{-\pi\omega/2\kappa} R_{\mathrm{II}}^{\mathrm{out}} \\ f_2^* &= e^{-\pi\omega/2\kappa}\rho_1 = e^{-\pi\omega/2\kappa} R_{\mathrm{I}}^{\mathrm{out}} + e^{\pi\omega/2\kappa} R_{\mathrm{II}}^{\mathrm{out}*} \end{aligned} \tag{39}$$

and

$$\begin{aligned} f_1 &= e^{\pi\omega/2\kappa} R_{\mathrm{I}}^{\mathrm{out}} + e^{-\pi\omega/2\kappa} R_{\mathrm{II}}^{\mathrm{out}*} \\ f_2 &= e^{-\pi\omega/2\kappa} R_{\mathrm{I}}^{\mathrm{out}*} + e^{\pi\omega/2\kappa} R_{\mathrm{II}}^{\mathrm{out}} \end{aligned} \tag{40}$$

They also form a complete orthonormal basis for the outgoing wave. Every outgoing wave packet crossing the event horizon may be expanded by them as

$$\phi^{\text{out}} = \int d\omega\, [2\sinh(\pi\omega/\kappa)]^{-1/2}(d_\omega^{(1)} f_1 + d_\omega^{(2)} f_2 + d_\omega^{(1)\dagger} f_1^* + d_\omega^{(2)\dagger} f_2^*) \quad (41)$$

$d_\omega^{(1)}$, $d_\omega^{(2)}$, $d_\omega^{(1)\dagger}$, $d_\omega^{(2)\dagger}$ are annihilation operators and creation operators, respectively. With them, one can construct another Fock space whose vacuum state is $|0\rangle'_{\text{out}}$:

$$d_\omega^{(1)}|0\rangle'_{\text{out}} = d_\omega^{(2)}|0\rangle'_{\text{out}} = 0 \quad (42)$$

It is easy to see that the ingoing wave solutions (22) form another complete orthonormal basis for the ingoing wave. Every ingoing wave packet can be expanded by them as

$$\phi^{\text{in}} = \int d\omega\, (a_\omega R^{\text{in}} + a_\omega^\dagger R^{\text{in}*}) \quad (43)$$

so one can define the ingoing vacuum state $|0\rangle_{\text{in}}$ as

$$a_\omega |0\rangle_{\text{in}} = 0 \quad (44)$$

Because the analytical properties of the modes (40) are the same as the modes (22) on the space-time manifold, and they are analytical not only inside but also outside the event horizon including the horizon itself, they should correspond to the same vacuum state, so

$$|0\rangle'_{\text{out}} = |0\rangle_{\text{in}} \quad (45)$$

It is easy to get the Bogoliubov transformation

$$\begin{aligned} b_\omega^{(1)} &= [2\sinh(\pi\omega/\kappa)]^{-1/2}[e^{\pi\omega/2\kappa} d_\omega^{(1)} + e^{-\pi\omega/2\kappa} d_\omega^{(2)\dagger}] \\ b_\omega^{(2)} &= [2\sinh(\pi\omega/\kappa)]^{-1/2}[e^{\pi\omega/2\kappa} d_\omega^{(2)} + e^{-\pi\omega/2\kappa} d_\omega^{(1)\dagger}] \end{aligned} \quad (46)$$

and to prove that there exist outgoing particles, in the ingoing vacuum state, as

$$_{\text{in}}\langle 0| b_\omega^{(1,2)\dagger} b_\omega^{(1,2)} |0\rangle_{\text{in}} = [2\sinh(\pi\omega/\kappa)]^{-1}\, e^{-\pi\omega/\kappa} = \frac{1}{e^{2\pi\omega/\kappa} - 1} \quad (47)$$

This shows that the ingoing vacuum state is transformed into an outgoing state containing thermal particles after scattering on the event horizon.

We obtain the Unruh-type analytical functions (39) and (40) making use of the analytical continuation method suggested by Damour and Ruffini, and we prove that there exists Hawking radiation by means of the Unruh-type Bogoliubov transformation.

3. FROM UNRUH TO DAMOUR–RUFFINI

Now, we will point out that the Unruh-type analytical function (40) can be directly obtained by means of the Damour–Ruffini analytical continuation, but through the upper half complex r plane. And we can get the Hawking thermal spectrum after calculating the inner product of the Unruh-type wave function by means of the Damour–Ruffini approach, without the Bogoliubov transformation (46).

Extending the outgoing waves $R_{\rm I}^{\rm out}$ and $R_{\rm I}^{\rm out*}$ outside the horizon to its inside by analytical continuation through the upper half complex r plane as

$$(r-2M) \rightarrow |r-2M|\, e^{i\pi} = (2M-r)\, e^{i\pi} \tag{48}$$

we have

$$R_{\rm I}^{\rm out} \rightarrow \tilde{\tilde{R}}_{\rm II}^{\rm out} = e^{-\pi\omega/\kappa} R_{\rm II}^{\rm out*} \tag{49}$$

$$R_{\rm I}^{\rm out*} \rightarrow \tilde{\tilde{R}}_{\rm II}^{\rm out*} = e^{\pi\omega/\kappa} R_{\rm II}^{\rm out} \tag{50}$$

The analytical wave functions across the event horizon can be written as

$$\psi_1 \sim R_{\rm I}^{\rm out} + e^{-\pi\omega/\kappa} R_{\rm II}^{\rm out*} \tag{51}$$

$$\psi_2 \sim R_{\rm I}^{\rm out*} + e^{\pi\omega/\kappa} R_{\rm II}^{\rm out} \tag{52}$$

Equation (51) multiplied by $e^{\pi\omega/2\kappa}$ is just f_1 in equation (40). Equation (52) multiplied by $e^{-\pi\omega/2\kappa}$ is f_2. It is clear that the Unruh-type analytical functions have been obtained by means of the analytical continuation method suggested by Damour and Ruffini.

Carrying out the Damour–Ruffini approach continually, we get the Hawking thermal spectrum from the Unruh-type wave functions (51) and (52), without the Bogoliubov transformation. Introducing a factor $N_1\, e^{\pi\omega/\kappa}$, we can write equation (51) as

$$\psi_1 = N_1 (e^{\pi\omega/\kappa} R_{\rm I}^{\rm out} + R_{\rm II}^{\rm out*}) \tag{53}$$

where N_1 is a normalization factor. Calculating the inner product, we have

$$1 = (\psi_1, \psi_1) = N_1^2 (e^{2\pi\omega/\kappa} - 1) \tag{54}$$

so we get the Hawking thermal spectrum

$$N_1^2 = \frac{1}{e^{2\pi\omega/\kappa} - 1} \tag{55}$$

Similarly, multiplying by a normalization factor N_2, we can show equation (52) as

$$\psi_2 = N_2 (R_{\rm I}^{\rm out*} + e^{\pi\omega/\kappa} R_{\rm II}^{\rm out}) \tag{56}$$

Its inner product is

$$1 = (\psi_2, \psi_2) = N_2^2(-1 + e^{2\pi\omega/\kappa}) \tag{57}$$

The thermal spectrum is still

$$N_2^2 = \frac{1}{e^{2\pi\omega/\kappa} - 1} \tag{58}$$

By the way, the difference between equations (15) and (34) is only the normalization factor N_ω. The thermal spectrum (20) obtained by means of the Damour–Ruffini approach is just that given by the inner product of ρ_1. Introducing a normalization factor, we can write as

$$\rho_1 = N_3(R_{\rm I}^{\rm out} + e^{\pi\omega/\kappa} R_{\rm II}^{\rm out*}) \tag{59}$$

Calculating the inner product

$$(\rho_1, \rho_1) = N_3^2(1 - e^{2\pi\omega/\kappa}) < 0 \tag{60}$$

so $(\rho_1, \rho_1) = -1$, we have the thermal spectrum

$$N_3^2 = \frac{1}{e^{2\pi\omega/\kappa} - 1} \tag{61}$$

Similarly, introducing a factor $N_4\, e^{\pi\omega/\kappa}$, we have

$$\rho_2 = N_4(e^{\pi\omega/\kappa} R_{\rm I}^{\rm out*} + R_{\rm II}^{\rm out}) \tag{62}$$

so we get

$$-1 = (\rho_2, \rho_2) = N_4^2(-e^{2\pi\omega/\kappa} + 1) \tag{63}$$

$$N_4^2 = \frac{1}{e^{2\pi\omega/\kappa} - 1} \tag{64}$$

With the Schwarzschild coordinates, neither $R_{\rm I}^{\rm out}$ nor $R_{\rm II}^{\rm out}$ is analytic at the event horizon. One can think that the black hole space-time is divided into two parts by the horizon, $r > r_H$ and $r < r_H$. When r is regarded as a complex number, both $R_{\rm I}^{\rm out}$ and $R_{\rm I}^{\rm out*}$ can be extended to the inside of the black hole by analytical continuation through the complex r plane, so each of the wave functions ψ_1, ψ_2, ρ_1, and ρ_2 can be regarded as an analytical function of the complex manifold. They are pure quantum states in the complex space-time. When r is regarded as a real number, the real space-time is divided into two parts by the event horizon. Each one of the outgoing waves ψ_1, ψ_2, ρ_1, and ρ_2 is also divided into two parts. One part is inside the black hole, another part is outside the black hole. No information can transit the event horizon from inner to outer, so the two parts of every outgoing wave are not correlated with each other. Then, the pure state wave functions ψ_1, ψ_2, ρ_1, and ρ_2 in the complex space-time are mixtures in the real space time, which shows thermal properties. This is the physical significance of the Damour–Ruffini approach.

4. CONCLUSION

The Unruh-type analytical wave functions can be obtained by means of the Damour–Ruffini approach. f_1^* and f_2^* (i.e., ρ_1 and ρ_2) given by the analytical continuation through the lower half complex r plane (Damour–Ruffini) are, respectively, the complex conjugate functions of f_1 and f_2 (i.e., ψ_1 and ψ_2) given by the continuation through the upper half complex r plane (Unruh). Normalizing these wave functions, or making use of them to construct the Bogoliubov transformation, one can get the same Hawking radiation spectrum.

The physical significance of the Damour–Ruffini approach is that the pure state wave function defined on the connected complex (r) space-time manifold is a mixture showing thermal properties in the real r space-time manifold, which is divided into two parts by the event horizon.

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